



Robotics 1

Minimal representations of orientation (Euler and roll-pitch-yaw angles) Homogeneous transformations

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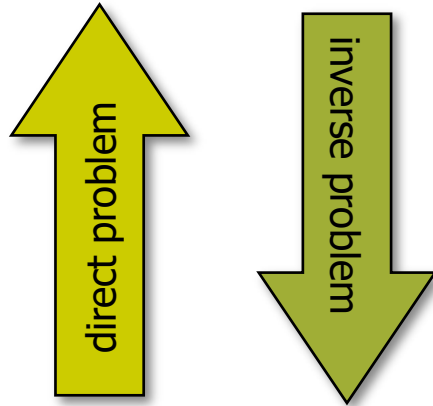
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"Minimal" representations

- rotation matrices: 9 elements
 - 3 orthogonality relationships
 - 3 unitary relationships

= 3 independent variables



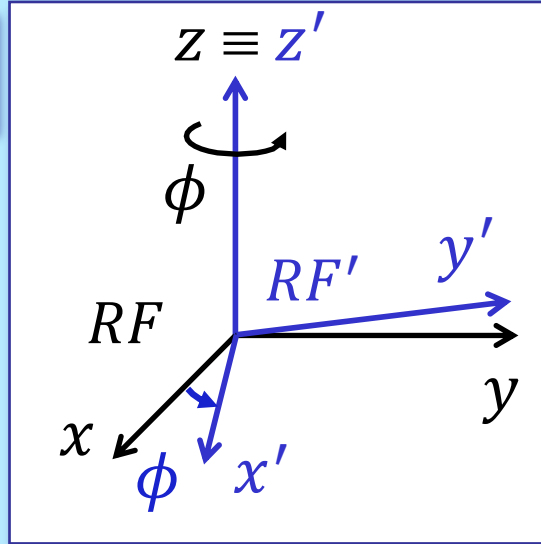
- sequence of 3 rotations w.r.t. independent axes
 - by angles $\alpha_i, i = 1, 2, 3$, around fixed (a_i) or moving/current (a'_i) axes
 - generically called Roll-Pitch-Yaw (fixed axes) or Euler (moving axes) angles
 - 12 + 12 possible different sequences (e.g., XYZ)
 - without contiguous repetitions of axes (e.g., no XXZ nor YZ'Z')
 - actually, only 12 sequences are different since we shall see that

$$\{(a_1, \alpha_1), (a_2, \alpha_2), (a_3, \alpha_3)\} \equiv \{(a'_3, \alpha_3), (a'_2, \alpha_2), (a'_1, \alpha_1)\}$$



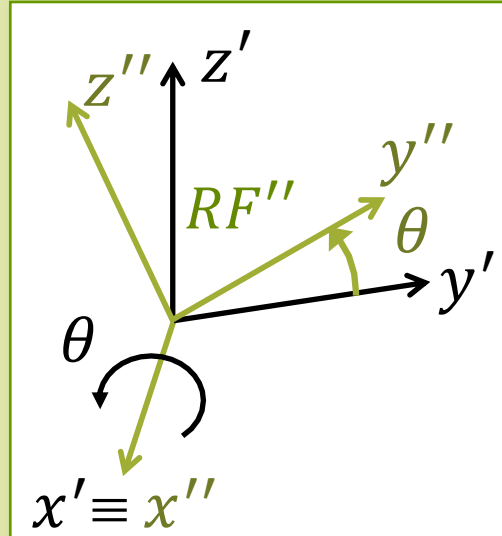
ZX'Z'' Euler angles

1



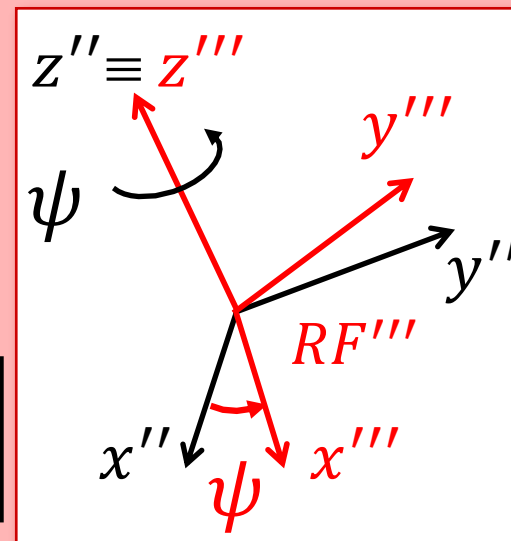
$$R_Z(\phi) = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2



$$R_{X'}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

3



$$R_{Z''}(\psi) = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$ZX'Z''$ Euler angles

- **direct problem:** given ϕ, θ, ψ , find R

$$R_{ZX'Z''}(\phi, \theta, \psi) = R_Z(\phi)R_{X'}(\theta)R_{Z''}(\psi)$$

order of definition
in concatenation $\rightarrow$

$$= \begin{bmatrix} c\phi c\psi - s\phi c\theta s\psi & -c\phi s\psi - s\phi c\theta c\psi & s\phi s\theta \\ s\phi c\psi + c\phi c\theta s\psi & -s\phi s\psi + c\phi c\theta c\psi & -c\phi s\theta \\ s\theta s\psi & s\theta c\psi & c\theta \end{bmatrix}$$

- given a vector $v''' = (x''', y''', z''')$ expressed in RF''' , its expression in the coordinates of RF is

$$v = R_{ZX'Z''}(\phi, \theta, \psi)v'''$$

- the orientation of RF''' is the **same** that would be obtained with the sequence of rotations

ψ around z , θ around x (**fixed**), ϕ around z (**fixed**)



ZX'Z'' Euler angles

- **inverse problem:** given $R = \{r_{ij}\}$, find ϕ, θ, ψ

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\phi c\psi - s\phi c\theta s\psi & -c\phi s\psi - s\phi c\theta c\psi & s\phi s\theta \\ s\phi c\psi + c\phi c\theta s\psi & -s\phi s\psi + c\phi c\theta c\psi & -c\phi s\theta \\ s\theta s\psi & s\theta c\psi & c\theta \end{bmatrix}$$

- $r_{13}^2 + r_{23}^2 = s^2\theta, r_{33} = c\theta \Rightarrow$

$$\theta = \text{atan2} \left\{ \pm \sqrt{r_{13}^2 + r_{23}^2}, r_{33} \right\}$$

two values differing just for the sign

- if $r_{13}^2 + r_{23}^2 \neq 0$ (i.e., $s\theta \neq 0$)

$$r_{31}/s\theta = s\psi, r_{32}/s\theta = c\psi \Rightarrow \psi = \text{atan2}\{r_{31}/s\theta, r_{32}/s\theta\}$$

- similarly...

$$\phi = \text{atan2}\{r_{13}/s\theta, -r_{23}/s\theta\}$$

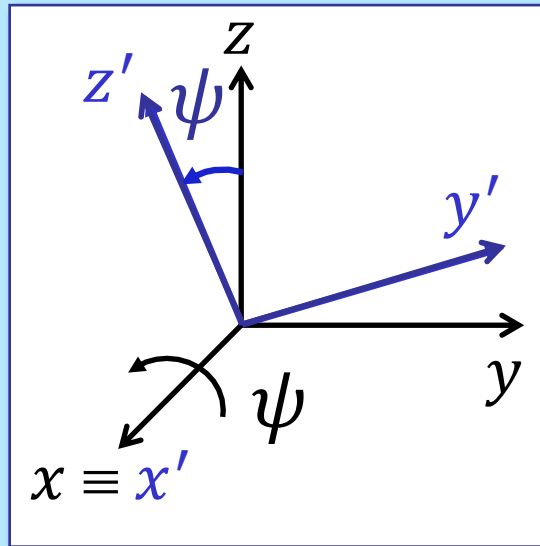
- there is always a **pair** of solutions in the regular case
- there are always **singularities** (here $\theta = 0$ **or** $\pm\pi$) $\Rightarrow$ only the **sum** $\phi + \psi$ **or** the **difference** $\phi - \psi$ can be determined



Roll-Pitch-Yaw angles (fixed XYZ)

1

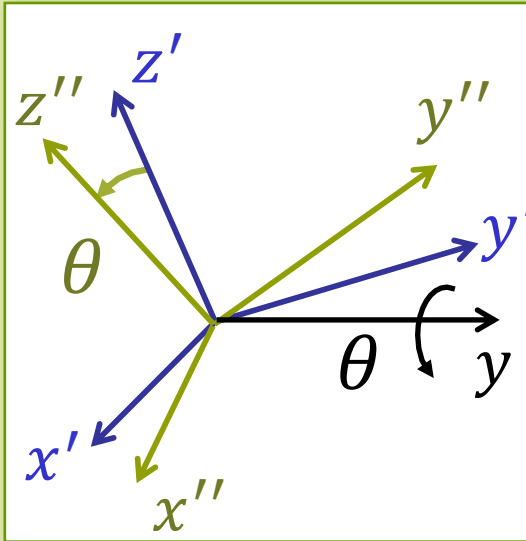
ROLL



$$R_X(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & -\sin \psi \\ 0 & \sin \psi & \cos \psi \end{bmatrix}$$

$$C_2 R_Z(\phi) C_2^T$$

$$\text{with } R_Z(\phi) = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



PITCH

2

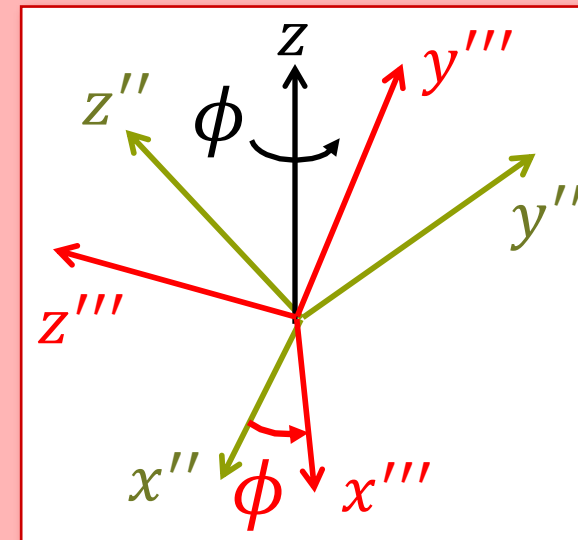
$$C_1 R_Y(\theta) C_1^T$$

with $R_Y(\theta) =$

$$\begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

3

YAW





Roll-Pitch-Yaw angles (fixed XYZ)

- **direct problem:** given ψ, θ, ϕ , find R

$$R_{RPY}(\psi, \theta, \phi) = R_Z(\phi)R_Y(\theta)R_X(\psi) \quad \Leftarrow \text{note the order of products!}$$

order of definition $\rightarrow$

$$= \begin{bmatrix} c\phi c\theta & c\phi s\theta s\psi - s\phi c\psi & c\phi s\theta c\psi + s\phi s\psi \\ s\phi c\theta & s\phi s\theta s\psi + c\phi c\psi & s\phi s\theta c\psi - c\phi s\psi \\ -s\theta & c\theta s\psi & c\theta c\psi \end{bmatrix}$$

- **inverse problem:** given $R = \{r_{ij}\}$, find ψ, θ, ϕ

- $r_{32}^2 + r_{33}^2 = c^2\theta, \quad r_{31} = -s\theta \Rightarrow$

$$\theta = \text{atan2} \left\{ -r_{31}, \pm \sqrt{r_{32}^2 + r_{33}^2} \right\}$$

for $r_{31} < 0$, two symmetric values w.r.t. $\pi/2$

- if $r_{32}^2 + r_{33}^2 \neq 0$ (i.e., $c\theta \neq 0$)

$$r_{32}/c\theta = s\psi, \quad r_{33}/c\theta = c\psi \Rightarrow$$

$$\psi = \text{atan2}\{r_{32}/c\theta, r_{33}/c\theta\}$$

- similarly ...

$$\phi = \text{atan2}\{r_{21}/c\theta, r_{11}/c\theta\}$$

- **singularities** for $\theta = \pm \pi/2 \Rightarrow$ only $\phi + \psi$ or $\phi - \psi$ are defined



...why this order in the product?

$$R_{RPY}(\psi, \theta, \phi) = R_Z(\phi) R_Y(\theta) R_X(\psi)$$

order of definition

“reverse” order in the product
(pre-multiplication...)

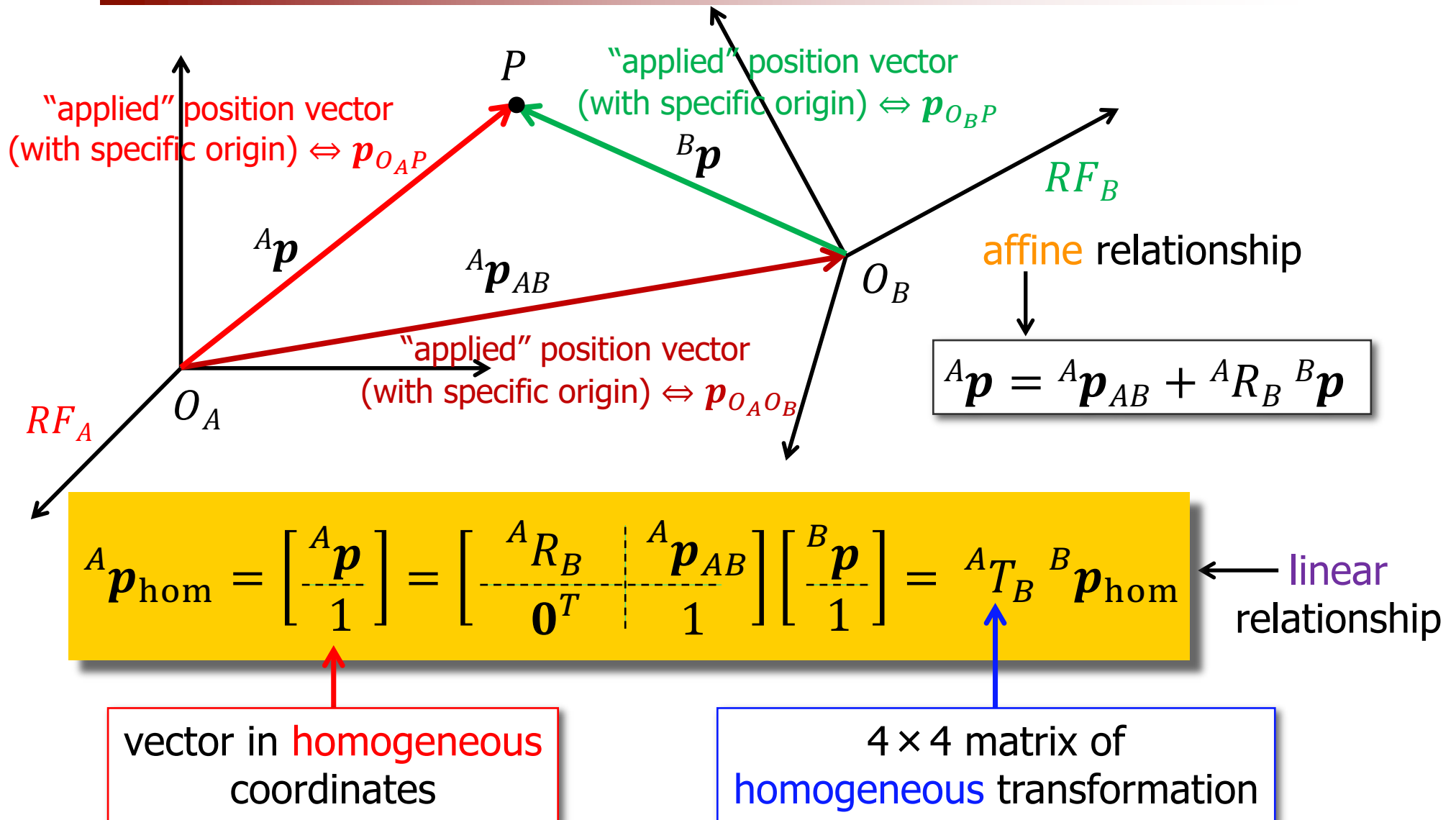
- need to refer each rotation in the sequence to one of the original **fixed** axes
 - use the angle/axis technique for each rotation in the sequence:
 $C R(\alpha) C^T$, with C being the rotation matrix **reverting** the previously made rotations (= “go back” to the original axes)

concatenating three rotations: $[] [] []$ (post-multiplication...)

$$\begin{aligned} R_{RPY}(\psi, \theta, \phi) &= [R_X(\psi)] [R_X^T(\psi) R_Y(\theta) R_X(\psi)] \\ &\quad [R_X^T(\psi) R_Y^T(\theta) R_Z(\phi) R_Y(\theta) R_X(\psi)] \\ &= R_Z(\phi) R_Y(\theta) R_X(\psi) \end{aligned}$$



Homogeneous transformations



Use of homogeneous transformation T



- describes the relation between two reference frames (relative **pose** = position & orientation)
- transforms the representation of a **position** vector (**applied** vector starting from the **origin** of the frame) from one frame to another frame
- it is a **roto-translation** operator on vectors in the three-dimensional space
- it is always invertible $\left({}^A T_B\right)^{-1} = {}^B T_A$
- can be composed, i.e., ${}^A T_B {}^B T_C = {}^A T_C$ ← note: it does not commute in general!



Affine vs linear computations

whiteboard...

$${}^1p = {}^1p_{01} + {}^1R_0 {}^0p$$

$${}^2p = {}^2p_{12} + {}^2R_1 {}^1p = {}^2p_{12} + {}^2R_1 {}^1p_{01} + {}^2R_1 {}^1R_0 {}^0p$$

$${}^3p = {}^3p_{23} + {}^3R_2 {}^2p = \dots = {}^2p_{23} + {}^3R_2 {}^2p_{12} + {}^3R_2 {}^2R_1 {}^1p_{01} + {}^3R_2 {}^2R_1 {}^1R_0 {}^0p$$

$${}^4p = {}^4p_{34} + {}^4R_3 {}^3p = \dots \quad \text{heavy on notation (and not only!)}$$

$${}^1T_0 = \begin{bmatrix} {}^1R_0 & {}^1p_{01} \\ 0^T & 1 \end{bmatrix} \Rightarrow {}^1p_{hom} = {}^1T_0 {}^0p_{hom}$$

$${}^2T_1 = \begin{bmatrix} {}^2R_1 & {}^2p_{12} \\ 0^T & 1 \end{bmatrix} \Rightarrow {}^2p_{hom} = {}^2T_1 {}^1T_0 {}^0p_{hom} = {}^2T_0 {}^0p_{hom}$$

$${}^3T_2 = \begin{bmatrix} {}^3R_1 & {}^3p_{23} \\ 0^T & 1 \end{bmatrix} \Rightarrow {}^3p_{hom} = {}^3T_2 {}^2T_1 {}^1T_0 {}^0p_{hom} = {}^3T_0 {}^0p_{hom}$$

$${}^4T_3 = \begin{bmatrix} {}^4R_3 & {}^4p_{34} \\ 0^T & 1 \end{bmatrix} \Rightarrow {}^4p_{hom} = {}^4T_3 {}^3T_2 {}^2T_1 {}^1T_0 {}^0p_{hom} = {}^4T_0 {}^0p_{hom}$$

Inverse of a homogeneous transformation



exchange $A \rightleftharpoons B$

... with the original vectors/matrices ...

$${}^A\mathbf{p} = {}^A\mathbf{p}_{AB} + {}^AR_B {}^B\mathbf{p} \quad {}^B\mathbf{p} = {}^B\mathbf{p}_{BA} + {}^BR_A {}^A\mathbf{p} = -{}^AR_B^T {}^A\mathbf{p}_{AB} + {}^AR_B^T {}^A\mathbf{p}$$



$$\begin{bmatrix} {}^AR_B & | & {}^A\mathbf{p}_{AB} \\ \hline 0 & 0 & 0 & | & 1 \end{bmatrix}$$

$$\begin{bmatrix} {}^BR_A & | & {}^B\mathbf{p}_{BA} \\ \hline 0 & 0 & 0 & | & 1 \end{bmatrix}$$

=

$$\begin{bmatrix} {}^AR_B^T & | & -{}^AR_B^T {}^A\mathbf{p}_{AB} \\ \hline 0 & 0 & 0 & | & 1 \end{bmatrix}$$

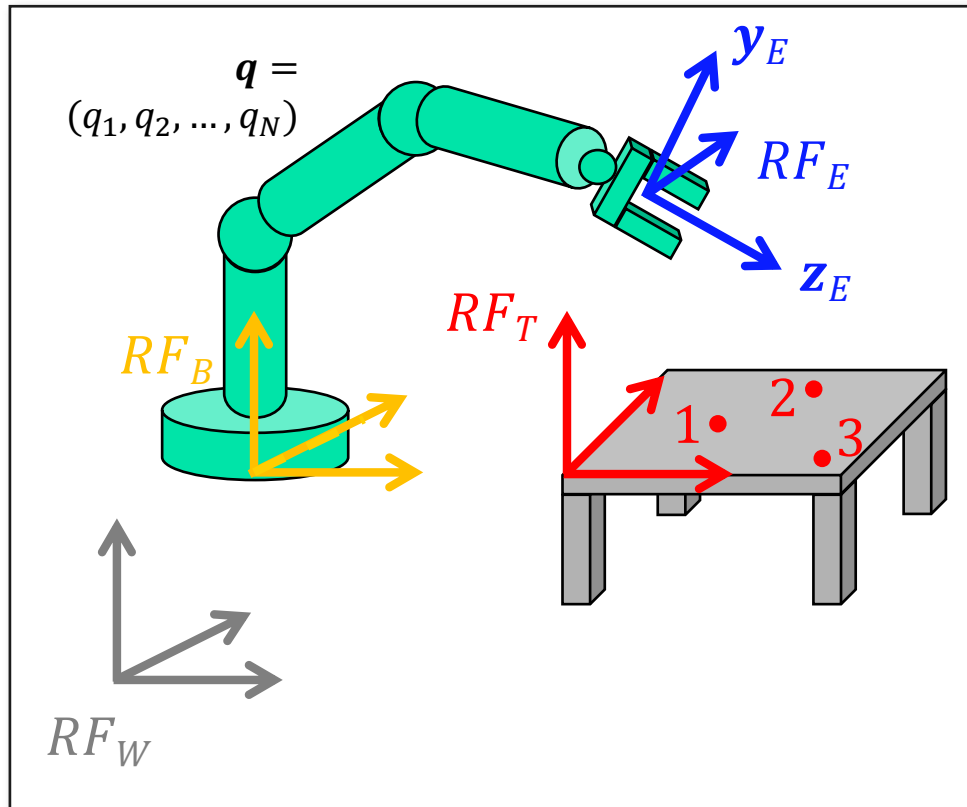
AT_B

BT_A

$({}^AT_B)^{-1}$



Defining a robotic task



absolute definition
of task

task definition relative
to the robot end-effector

$${}^W T_T = {}^W T_B {}^B T_E {}^E T_T$$

known, once
the robot
is placed

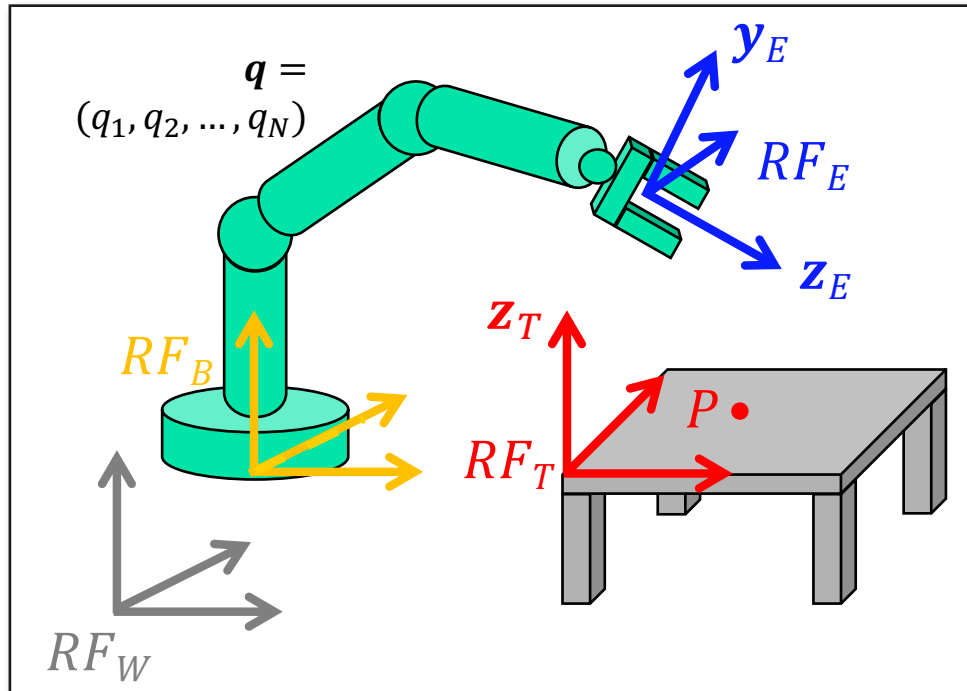
direct kinematics of the
robot arm (function of q)

solve for q
(inverse
kinematics)

$${}^B T_E(q) = {}^W T_B^{-1} {}^W T_T {}^E T_T^{-1} = \text{constant}$$



Example of task definition



Q: where is the EE frame w.r.t. the table frame?

$${}^T T_E = \begin{bmatrix} {}^T R_E & {}^T \mathbf{p}_{TE} \\ 0^T & 1 \end{bmatrix} = {}^E T_T^{-1}$$

with ${}^T R_E = ({}^E R_T)^T = {}^E R_T$

$${}^T \mathbf{p}_{TE} = {}^T \mathbf{p} - {}^T R_E {}^E \mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ h \end{bmatrix}$$

- the robot carries a **depth camera** (e.g., a Kinect) on the end-effector
- the end-effector should go to a pose above the point P on the table, pointing its approach axis $\mathbf{z}_E$ **downward** and being **aligned** with the table sides

$${}^E R_T = \begin{bmatrix} {}^E \mathbf{x}_T & {}^E \mathbf{y}_T & {}^E \mathbf{z}_T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- point P is known in the table frame RF_T

$${}^T \mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ 0 \end{bmatrix}$$

- the robot proceeds by **centering point P** in its camera image until it senses a **depth h** from the table (in RF_E)

$${}^E \mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix}$$

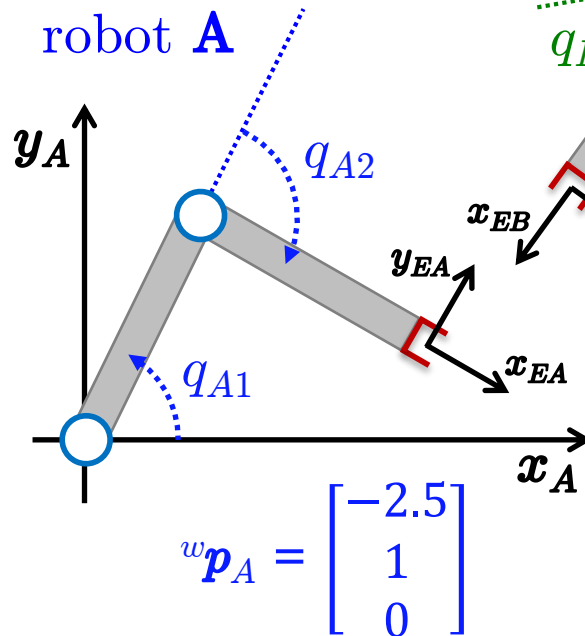


A robotic problem with T matrices

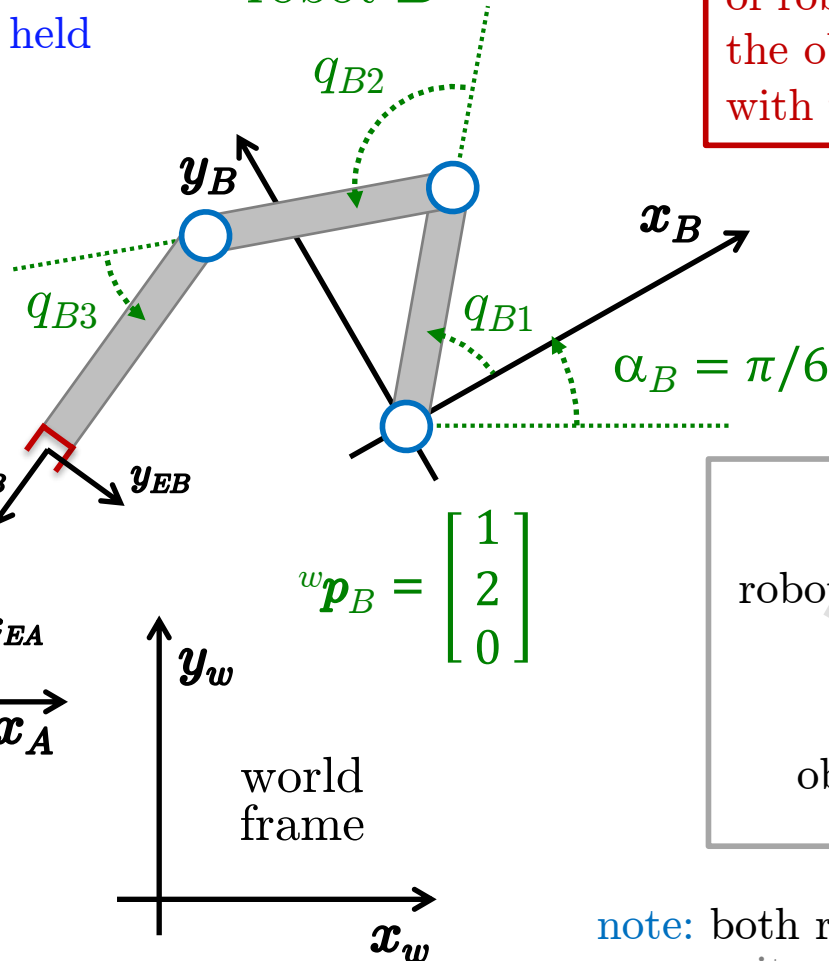
Task: 2R planar robot **A** should hand over an object at a given location to 3R planar robot **B**

configuration of robot **A**
with which the object is being held

$$\mathbf{q}_A = \begin{bmatrix} \pi/3 \\ -\pi/2 \end{bmatrix}$$

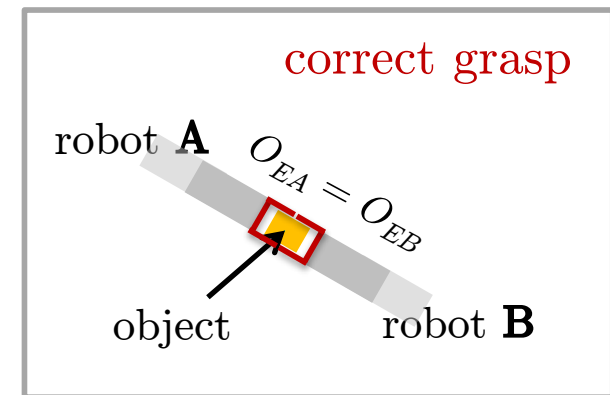


robot **B**



Q: Find a configuration $\mathbf{q}_B$ of robot **B** so as to grasp the object held by robot **A** with the right orientation

Ex #3, Robotics 1
exam of Sep 11, 2020



note: both robots have
unitary link lengths



Solution procedure

$${}^wT_A = \begin{pmatrix} {}^wR_A & {}^wP_A \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} I_{3 \times 3} & \begin{matrix} -2.5 \\ 1 \\ 0 \end{matrix} \\ \mathbf{0}^T & 1 \end{pmatrix} \quad \text{base frame of robot } \mathbf{A} \text{ w.r.t. world}$$

$${}^wT_B = \begin{pmatrix} {}^wR_B & {}^wP_B \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos \alpha_B & -\sin \alpha_B & 0 & 1 \\ \sin \alpha_B & \cos \alpha_B & 0 & 2 \\ 0 & 0 & 1 & 0 \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} 0.8660 & -0.5 & 0 & 1 \\ 0.5 & 0.8660 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ \mathbf{0}^T & 1 \end{pmatrix} \quad \text{base frame of robot } \mathbf{B} \text{ w.r.t. world}$$

$$\begin{aligned} {}^AT_{EA} &= \begin{pmatrix} {}^AR_{EA} & {}^AP_{EA} \\ \mathbf{0}^T & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(q_{A1} + q_{A2}) & -\sin(q_{A1} + q_{A2}) & 0 & \cos q_{A1} + \cos(q_{A1} + q_{A2}) \\ \sin(q_{A1} + q_{A2}) & \cos(q_{A1} + q_{A2}) & 0 & \sin q_{A1} + \sin(q_{A1} + q_{A2}) \\ 0 & 0 & 1 & 0 \\ \mathbf{0}^T & 1 \end{pmatrix} \quad \begin{array}{l} \text{end-effector frame of robot } \mathbf{A} \\ \text{w.r.t. its base frame (uses } \mathbf{q}_A) \\ = \text{direct kinematics of robot } \mathbf{A}! \end{array} \\ &= \begin{pmatrix} 0.8660 & 0.5 & 0 & 1.3660 \\ -0.5 & 0.8660 & 0 & 0.3660 \\ 0 & 0 & 1 & 0 \\ \mathbf{0}^T & 1 \end{pmatrix} \end{aligned}$$

$${}^{EA}T_{EB} = \begin{pmatrix} {}^{EA}R_{EB} & {}^{EA}P_{EB} \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mathbf{0}^T & 1 \end{pmatrix} \quad \begin{array}{l} \text{end-effector frame of robot } \mathbf{B} \\ \text{w.r.t. end-effector frame of robot } \mathbf{A} \text{ to} \\ \text{realize the right grasp for correct handover} \end{array}$$



Solution procedure

$${}^wT_A {}^AT_{EA} {}^EA T_{EB} = {}^wT_B {}^BT_{EB}$$

kinematic equation defining the task

end-effector frame of robot **B**
w.r.t. world **passing via the**
given configuration of robot A

=

end-effector frame of robot **B**
w.r.t. world **passing via its**
base frame

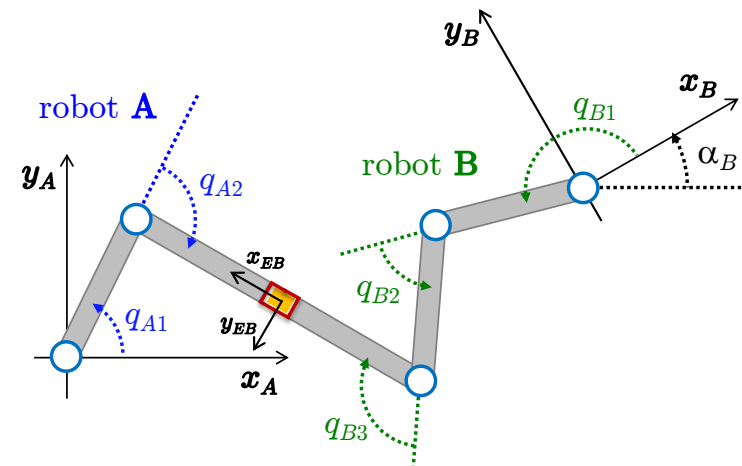
$${}^BT_{EB,d} = \begin{pmatrix} {}^BR_{EB,d} & {}^Bp_{EB,d} \\ \mathbf{0}^T & 1 \end{pmatrix} = ({}^wT_B)^{-1} {}^wT_A {}^AT_{EA} {}^EA T_{EB}$$

$$= \begin{pmatrix} -0.5 & -0.8660 & 0 & -2.1651 \\ 0.8660 & -0.5 & 0 & 0.5179 \\ 0 & 0 & 1 & 0 \\ \mathbf{0}^T & & & 1 \end{pmatrix} =$$

desired end-effector frame
of robot **B** w.r.t. its base
= input for the
inverse kinematics of robot B!

one solution $\mathbf{q}_B$ (out of 2) of the
inverse kinematics of robot B

$$\mathbf{q}_B = \begin{bmatrix} q_{B1} \\ q_{B2} \\ q_{B3} \end{bmatrix} = \begin{bmatrix} 2.7939 \\ 1.1076 \\ -1.8071 \end{bmatrix} [\text{rad}] = \begin{bmatrix} 160.08^\circ \\ 63.46^\circ \\ -103.54^\circ \end{bmatrix}$$





Remarks on homogeneous matrices

- the main tool used for computing the **direct kinematics** of robot manipulators
- relevant in many other applications (in robotics and beyond)
 - in positioning/orienting a vision camera (matrix bT_c with extrinsic parameters of the camera pose)
 - in computer graphics, for the real-time visualization of 3D solid objects when changing the observation point

$${}^AT_B = \begin{bmatrix} {}^AR_B & {}^Ap_{AB} \\ \alpha_x & \sigma \\ \alpha_y & \\ \alpha_z & \end{bmatrix}$$

all zero
in robotics

coefficients of
perspective
deformation

scaling
coefficient

always unitary
in robotics